ESTIMATES FOR THE CORONA THEOREM ON $H^\infty_{\mathbb{I}}(\mathbb{D})$

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Abstract. Let $\mathbb{I}$ be a proper ideal of $H^\infty(\mathbb{D})$. We prove the corona theorem for infinitely many generators in the algebra $H^\infty_{\mathbb{I}}(\mathbb{D})$. This extends the finite corona results of Mortini, Sasane, and Wick [8]. We also provide the estimates for corona solutions. Moreover, we prove a generalized Wolff's Ideal Theorem for this sub-algebra.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be an open unit disk in the complex plane $\mathbb{C}$ and $H^\infty(\mathbb{D})$ be the set of all bounded analytic functions with the norm $\|f\|_\infty = \sup\limits_{z\in\mathbb{D}}|f(z)| < \infty$. In 1962, Carleson proved his famous corona theorem which states that the ideal, $\mathcal{I}$, generated by a finite set of functions $\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ is the entire space $H^\infty(\mathbb{D})$, if for some $\epsilon > 0$, $\sum_{i=1}^n |f_i(z)|^2 \geq \epsilon$ for all $z \in \mathbb{D}$. In 1979, Wolff gave a simplified proof of Carleson's corona theorem, which can be found in [5], that made use of H^2-Carleson's measures and Littlewood-Paley expressions. Both Carleson and Wolff provided the bounds for corona solutions depending on the number of functions n. Later, Rosenblum [14], Tolokonnokov [20] , and Uchiyama [26], independently, extended the corona theorem for infinitely many functions, where as the best estimate for the corona solution was due to Uchiyama as follows:

Corona Theorem. *Let* $\{f_i\}_{i=1}^\infty \subset H^\infty(\mathbb{D})$, *with*

$$0 < \epsilon^2 \leq \sum_{i=1}^\infty |f_i(z)|^2 \leq 1 \textit{ for all } \quad z \in \mathbb{D}.$$

Then there exist $\{g_i\}_{i=1}^\infty \subset H^\infty(\mathbb{D})$ *such that*

$$\sum_{i=1}^\infty f_i(z)g_i(z) = 1 \quad \textit{for all} \quad z \in \mathbb{D}$$

2010 *Mathematics Subject Classification.* Primary: 30H50; Secondary: 30H80, 46J20.

Key words and phrases. corona theorem, Wolff's theorem, $H^\infty(\mathbb{D})$, ideals.

and

$$\sup_{z\in\mathbb{D}}\{\sum_{i=1}^{\infty}|g_i(z)|^2\}\leq\frac{9}{\epsilon^2}\ln\frac{1}{\epsilon^2},\quad \textit{for}\ \epsilon^2<\frac{1}{e}.$$

The main purpose of this paper is to extend the corona theorem for infinitely many functions in $H_{\mathbb{I}}^{\infty}(\mathbb{D})$. Moreover, we provide the estimates for the corona solutions. This will completely settle the conjecture of Ryle [15].

The algebra, $H_{\mathbb{I}}^{\infty}(\mathbb{D})$, of our interest is defined as follows:
Let $\mathbb{I}$ be any proper closed ideal in $H^{\infty}(\mathbb{D})$, and define

$$H_{\mathbb{I}}^{\infty}(\mathbb{D}):=\{c+\phi\mid c\in\mathbb{C}\text{ and }\phi\in\mathbb{I}\}.$$

Then $H_{\mathbb{I}}^{\infty}(\mathbb{D})$ is a sub-algebra of $H^{\infty}(\mathbb{D})$. We regard $(H_{\mathbb{I}}^{\infty}(\mathbb{D}))_{l^2}$ as a sub-algebra of $H_{l^2}^{\infty}(\mathbb{D})$, where $H_{l^2}^{\infty}(\mathbb{D})$ is a sequence of bounded analytic functions. Also, for $F=(f_1,f_2,...)$, $f_j\in H^{\infty}(\mathbb{D})$, we use the norm

$$\|F\|_{\infty}=\sup_{z\in\mathbb{D}}\left(\sum_{i=1}^{\infty}|f_i(z)|^2\right)^{1/2}.$$

In [8], Mortini, Sasane, and Wick proved the corona theorem for finitely many generators in $H_{\mathbb{I}}^{\infty}(\mathbb{D})$. In fact, [8] provided the estimates on the solutions g_j in terms of the parameters ϵ and n (the number of functions f_j). In this paper, we prove an analogous result of Uchiyama for the sub-algebra $H_{\mathbb{I}}^{\infty}(\mathbb{D})$ by removing the dependency of estimates on n.

Let $f\in H_{\mathbb{I}}^{\infty}(\mathbb{D})$, say $f(z)=c+\phi(z)$, for $\phi\in\mathbb{I}$ and $c\in\mathbb{C}$. For simplicity, we use the notation: $f(z)=f_c+\phi_f(z)$, where $f_c\in\mathbb{C}$ and $\phi_f\in\mathbb{I}$. Similarly, let $F=(f_1,f_2,...)$, $f_j\in H_{\mathbb{I}}^{\infty}(\mathbb{D})$. Then for $z\in\mathbb{D}$, we write $F(z)=F_c+\phi_F(z)$.

We are now ready to state our Main Theorem, which extends to the corona theorem for infinitely many functions in $H_{\mathbb{I}}^{\infty}(\mathbb{D})$.

Theorem 1.1. *Let* $F(z)=(f_1(z),f_2(z),\dots)$, $f_j\in H_{\mathbb{I}}^{\infty}(\mathbb{D})$ *and*

$$0<\epsilon^2\leq F(z)F(z)^*\ \leq 1\ \textit{ for all }\ z\in\mathbb{D}.$$

Then there exists $U=(u_1(z),u_2(z),\dots)$, $u_j\in H_{\mathbb{I}}^{\infty}(\mathbb{D})$ *such that*

$$(a)\ F(z)U(z)^T=1\ \textit{ for all } z\in\mathbb{D}\ \textit{ and}$$

$$(b)\ \|U\|_{\infty}\leq\left(1+\frac{1}{\|F_c\|}\right)\frac{9}{\epsilon^2}\ln\left(\frac{1}{\epsilon^2}\right).$$

In order to generalize the corona theorem, it is natural to ask if the corona theorem still holds true if we replace the lower bound, ϵ, in the corona condition by any $H^\infty(\mathbb{D})$ functions. Namely, let $h, f_1, f_2, ..., f_n \in H^\infty(\mathbb{D})$ such that

$$|h(z)| \leq \sum_{i=1}^{n} |f_i(z)| \leq 1 \text{ for all } \quad z \in \mathbb{D}. \tag{1}$$

Then the question is does (1) always implies $h \in \mathcal{I}(f_1, f_2, ..., f_n)$, ideal generated by $f_1, f_2, ..., f_n$? Of course, (1) is a necessary condition, but the counter example provided by Rao [12] suggests that it is far from being sufficient.

Rao's Counter Example: If B_1 and B_2 are Blaschke products without common zeros for which $\inf_{z\in\mathbb{D}} (|B_1(z)| + |B_2(z)|) = 0$, then $|B_1B_2| \leq (|B_1|^2 + |B_2|^2)$, but $B_1B_2 \notin \mathcal{I}\left(B_1^2, B_2^2\right)$.

However, T. Wolff's beautiful proof (see [5], Theorem 2.3 in page 319) showed that the condition (1) is sufficient for $h^3 \in \mathcal{I}(f_1, f_2, ..., f_n)$. Wolff's Theorem can be rephrased as follows:

Wolff's Theorem. *Let* $F(z) = (f_1(z), f_2(z), \ldots, f_n(z)), f_j \in H^\infty(\mathbb{D}),\ h \in H^\infty(\mathbb{D})$. *If*

$$|h(z)| \leq \sqrt{F(z)F(z)^*} \ \textit{for all} \quad z \in \mathbb{D},$$

then

$$h^3 \in \mathcal{I}(\{f_j\}_{j=1}^n).$$

But, it was shown by Treil [21] that this is not sufficient for $p = 2$.

Many authors, independently, have considered this question, including Cegrell [2], Pau [11], Trent [23], and Treil [22], for $p = 1$. We refer this as a problem of "ideal membership." It is Treil who has given the best known sufficient condition for ideal membership. We state Treil's Theorem as follows:

Ideal Theorem (Treil). *Let* $F(z) = (f_1(z), f_2(z), ...),\ f_j \in H^\infty(\mathbb{D})$, $F(z)F(z)^* \leq 1$ *for all* $z \in \mathbb{D}$, *and* $h \in H^\infty(\mathbb{D})$ *such that*

$$F(z)F(z)^*\ \psi\left(F(z)F(z)^*\right) \geq |h(z)| \textit{ for all } z \in \mathbb{D},$$

where $\psi : [0, 1] \to [0, 1]$ *is a non-decreasing function such that* $\int_0^1 \frac{\psi(t)}{t} dt < \infty$. *Then there exists* $G \in H^\infty_{l^2}(\mathbb{D})$ *such that*

$$F(z)G(z)^T = h(z), \quad \textit{for all } \ z \in \mathbb{D}.$$

An example of a function ψ that works in the case when $F(z)$ is an n-tuple, $n < \infty$, is

$$\psi(t) = \frac{1}{(\ln t^{-2})(\ln_2 t^{-2}) \ldots (\ln_n t^{-2})(\ln_{n+1} t^{-2})^{1+\epsilon}},$$

where $\ln_k(t) = \underbrace{\ln \ln ... ln}_{\text{k+1 times}}(t)$ and $\epsilon > 0$.

Applying Treil's result, we extend the analogue of "ideal theorem" on $H^\infty_{\mathbb{I}}(\mathbb{D})$. Recall that $H^\infty_{\mathbb{I}}(\mathbb{D})$ is a sub-algebra of $H^\infty(\mathbb{D})$. Also, for $F = (f_1, f_2, ...), f_j = f_{c_j} + \phi_{f_j} \in H^\infty_{\mathbb{I}}(\mathbb{D})$, we denote $F = F_c + \phi_F$. In the case that $F_c = 0$, several authors have given sufficient conditions for ideal membership, for example, see [6], [7], and [13]. For the case $F_c \neq 0$, we provide the following theorem:

Theorem 1.2. *Let $F(z) = (f_1(z), f_2(z), \ldots),\ f_j \in H^\infty_{\mathbb{I}}(\mathbb{D})$ such that $F_c \neq 0$, and suppose*

$$|h(z)| \leq F(z)F(z)^* \psi\left(F(z)F(z)^*\right) \leq 1 \text{ for all } z \in \mathbb{D},$$

where ψ is the function given in Treil's theorem. Then there exists $V = (v_1(z), v_2(z), \ldots),\ v_j \in H^\infty_{\mathbb{I}}(\mathbb{D})$ such that

$$(a)\ F(z)V(z)^T = h(z) \text{ for all } z \in \mathbb{D} \text{ and}$$

$$(b)\ \|V\|_\infty \leq C_0 \left(1 + \frac{1}{\|F_c\|}\right),$$

where C_0 is the estimate for the $H^\infty(\mathbb{D})$ solution obtained in [22].

Corollary 1. *Let $F(z) = (f_1(z), f_2(z), \ldots),\ f_j \in H^\infty_{\mathbb{I}}(\mathbb{D})$ such that $F_c \neq 0$, and suppose*

$$|h(z)| \leq \sqrt{F(z)F(z)^*} \leq 1 \text{ for all } z \in \mathbb{D}.$$

Then there exists $V = (v_1(z), v_2(z), \ldots),\ v_j \in H^\infty_{\mathbb{I}}(\mathbb{D})$ such that

$$(a)\ F(z)V(z)^T = h^3(z) \text{ for all } z \in \mathbb{D} \text{ and}$$

$$(b)\ \|V\|_\infty \leq C_1 \left(1 + \frac{1}{\|F_c\|}\right),$$

where C_1 is the estimate for the $H^\infty(\mathbb{D})$ solution obtained in [23].

2. Preliminaries

In this section, we discuss the method of our proofs and also provide some required lemmas. To prove Theorem 1.1 and Theorem 1.2 in $H^\infty_{\mathbb{I}}(\mathbb{D})$, we first find the corresponding solutions in the bigger algebra $H^\infty(\mathbb{D})$. Then we add some correction terms on the $H^\infty(\mathbb{D})$ -solutions to get the required solutions in our smaller algebra $H^\infty_{\mathbb{I}}(\mathbb{D})$.

For example, provided the corona condition, using Uchiyama version of corona theorem, we can easily find a solution G in $(H^\infty(\mathbb{D}))_{l^2}$ such that $F(z)G(z)^T = 1$ for all $z \in \mathbb{D}$. But, our goal is finding a solution $U \in (H^\infty_{\mathbb{I}}(\mathbb{D}))_{l^2}$ such that $F(z)U(z)^T = 1$ for all $z \in \mathbb{D}$. For this, if we can find an operator Q so that $M_Q(H^\infty(\mathbb{D}))_{l^2} \subseteq (H^\infty(\mathbb{D}))_{l^2}$ and for all $z \in \mathbb{D}$, $\operatorname{ran} Q(z) = \ker F(z)$, then we can construct the required solution U as

$$U^T := G^T + QX^T,$$

with a right choice of $X \in (H^\infty(\mathbb{D}))_{l^2}$. This solves our problem as follows:

$$F(z)U(z)^T = F(z)G(z)^T = 1, \text{ for all } z \in \mathbb{D},$$

and the proper choice of X will make $U \in (H^\infty_{\mathbb{I}}(\mathbb{D}))_{l^2}$.

The next lemma is a linear algebra result which gives us the desired Q operator and so enables us to write down the most general pointwise solution of $F(z)U(z)^T = 1$. This lemma can be found in Ryle -Trent [16], but we provide a proof for convenience.

Lemma 2.1. *Let $\{a_j\}_{j=1}^\infty \in l^2$ and $A = (a_1, a_2, \dots) \in \mathcal{B}(l^2, \mathbb{C})$. Then there exists a matrix Q_A of order $\infty \times \infty$ such that the entries of Q_A are either $\underline{+}a_j$ or 0 and Q_A satisfies:*

$$\operatorname{ran} Q_A = \ker A \tag{2}$$

and

$$(AA^*)I_{l^2} - A^*A = Q_AQ_A^* \quad \textit{with} \quad \|Q_A\|_{\mathcal{B}(l^2)} \leq \|A\|_{l^2}.$$

Also, if $\{d_j\}_{j=1}^\infty \in l^2$ and $D = (d_1, d_2, \dots)$, then

$$(AD^T)I_{l^2} - D^TA = Q_AQ_D^T. \tag{3}$$

Following few examples should be helpful to understand the Lemma 2.1 in a simple way.

Let $f_1, f_2, ..., f_n \in H^\infty(\mathbb{D})$ and fix $z \in \mathbb{D}$. Take $F = [f_1 \;\; f_2 \;, ..., \; f_n]$. For $n = 2$, $F = [f_1 \;\; f_2]$, $Q_F = \begin{bmatrix} f_2 \\ -f_1 \end{bmatrix}$.

Thus, $(FF^*)I_2 - F^*F = \begin{bmatrix} |f_2|^2 & -\bar{f}_1 f_2 \\ \bar{f}_2 f_1 & |f_1|^2 \end{bmatrix} = Q_F Q_F^*$.

Also, for any $D = \begin{bmatrix} d_1 & d_2 \end{bmatrix}$,

$$(FD^T)I_2 - D^TF = \begin{bmatrix} f_2 d_2 & -d_1 f_2 \\ -d_2 f_1 & f_1 d_1 \end{bmatrix} = Q_F Q_D^T.$$

Similarly, for $n = 3$, we take $F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$.

So,

$$Q_F = \begin{bmatrix} f_2 & f_3 & 0 \\ -f_1 & 0 & f_3 \\ 0 & -f_1 & -f_2 \end{bmatrix}.$$

And, for $n = 4$, $F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}$ and

$$Q_F = \begin{bmatrix} f_2 & f_3 & f_4 & 0 & 0 & 0 \\ -f_1 & 0 & 0 & f_3 & f_4 & 0 \\ 0 & -f_1 & 0 & -f_2 & 0 & f_4 \\ 0 & 0 & -f_1 & 0 & -f_2 & f_3 \end{bmatrix}.$$

Form the above pattern, it is easy to see that the operators Q_F's can be constructed inductively. Also, it is clear from (3), applied to $A = F(z)$ and $Q_D = Q_{F(z)}$, that $\operatorname{ran} Q_F(z) = \ker F(z)$.

We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1. For $k \in \mathbb{N}$, define

$$A_k = \begin{bmatrix} 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ c_{k+1} & c_{k+2} & c_{k+3} & \dots \\ -c_k & 0 & 0 & \dots \\ 0 & -c_k & 0 & \dots \\ 0 & 0 & -c_k & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Multiplying A_k by A_k^*, we get

$$A_k A_k^* = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & \sum_{j=k+1}^{\infty} |c_j|^2 & -\bar{c}_k c_{k+2} & -\bar{c}_k c_{k+3} & \dots \\ 0 & \dots & 0 & -c_k \bar{c}_{k+2} & |c_k|^2 & 0 & \dots \\ 0 & \dots & 0 & -c_k \bar{c}_{k+3} & 0 & |c_k|^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Hence,

$$\sum_{k=1}^{\infty} A_k A_k^* = \begin{bmatrix} \sum_{k\neq 1}^{\infty} |c_k|^2 & -\bar{c}_1 c_2 & -\bar{c}_1 c_3 & \dots \\ -\bar{c}_2 c_1 & \sum_{k\neq 2}^{\infty} |c_k|^2 & -\bar{c}_2 c_3 & \dots \\ -\bar{c}_3 c_1 & -\bar{c}_3 c_2 & \sum_{k\neq 3}^{\infty} |c_k|^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = CC^* I_{l^2} - C^* C.$$

Thus the required operator Q_A can be defined as

$$Q_A = [A_1, A_2, ...] \in \mathcal{B}(\oplus_1^{\infty} l^2, l^2).$$

We note that (3) follows in a similar manner.

□

We also need the following key lemma.

Lemma 2.2. *Assume that* $\{f_j\}_{j=1}^{\infty} \subset H_{\mathbb{I}}^{\infty}(\mathbb{D})$ *and*

$$0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |f_j(z)|^2 \leq 1 \quad \textit{for all } z \in \mathbb{D}.$$

Then

$$\textit{(a)} \;\; \epsilon^2 \leq F_c F_c^\star = \sum_{j=1}^{\infty} |f_{c_j}|^2 \leq 1$$

and

$$\textit{(b)} \;\; \|\phi_F\|_\infty = \sup_{z\in\mathbb{D}} \left(\sum_{j=1}^{\infty} |\phi_{f_j}(z)|^2 \right) \leq 2.$$

Proof. Since for all $z \in \mathbb{D}$,

$$\epsilon^2 \leq \sum_{j=1}^{\infty} |f_{c_j} + \phi_{f_j}(z)|^2 \leq 1,$$

we have that for each $N \in \mathbb{N}$,

$$\sum_{j=1}^{N} |f_{c_j} + \phi_{f_j}(z)|^2 \leq 1.$$

But, $\{\phi_{f_j}\}_{j=1}^{N} \subset \mathbb{I}$ and $\mathbb{I}$ is a proper ideal, so by the corona theorem

$$\inf_{z\in\mathbb{D}} \sum_{j=1}^{N} |\phi_{f_j}(z)|^2 = 0.$$

This means that for each N

$$\sum_{j=1}^{N} |f_{c_j}|^2 \leq 1, \text{ and hence } \sum_{j=1}^{\infty} |f_{c_j}|^2 \leq 1.$$

Thus, (b) holds, since for $z \in \mathbb{D}$

$$\left(\sum_{j=1}^{\infty} |\phi_{f_j}(z)|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{\infty} |f_{c_j} + \phi_{f_j}(z)|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{\infty} |f_{c_j}|^2\right)^{\frac{1}{2}} \leq 2.$$

Now by the Rosenblum- Tolokonnikov-Uchiyama version of the corona theorem, since $\{\phi_{f_j}\}_{j=1}^{\infty} \subset \mathbb{I}$ and $\mathbb{I}$ is a proper closed ideal and $\sup\limits_{z\in\mathbb{D}} \sum_{j=1}^{\infty} |\phi_{f_j}(z)|^2 \leq 2 < \infty$, we have

$$\inf_{z\in\mathbb{D}} \sum_{j=1}^{\infty} |\phi_{f_j}(z)|^2 = 0.$$

Thus there exist $\{z_k\}_{k=1}^{\infty} \subset \mathbb{D}$ so that $\lim\limits_{k\to\infty} \sum_{j=1}^{\infty} |\phi_{f_j}(z_k)|^2 = 0$.

Therefore, from

$$\epsilon \leq \left(\sum_{j=1}^{\infty} |f_{c_j} + \phi_{f_j}(z_k)|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{\infty} |f_{c_j}|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{\infty} |\phi_{f_j}(z_k)|^2\right)^{\frac{1}{2}},$$

we deduce that

$$\epsilon^2 \leq \sum_{j=1}^{\infty} |f_{c_j}|^2.$$

So (a) follows. □

Now we are ready to prove our theorems.

3. The Proofs

Proof of Theorem 1.1. Let $F \in (H^{\infty}_{\mathbb{I}}(\mathbb{D}))_{l^2}$, and suppose

$$0 < \epsilon^2 \leq F(z)F(z)^* \leq 1 \text{ for all } z \in \mathbb{D}.$$

Then we know that there is a corona solution for F, say G, which lies in $(H^{\infty}(\mathbb{D}))_{l^2}$ such that

$$F(z)G(z)^T = 1, \text{ for all } z \in \mathbb{D} \text{ and}$$

$$\|G\|_{\infty} \leq \frac{9}{\epsilon^2} \ln\left(\frac{1}{\epsilon^2}\right).$$

Our aim is finding $U \in (H^\infty_{\mathbb{I}}(\mathbb{D}))_{l^2}$ such that $F(z)U(z)^T = 1$ for all $z \in \mathbb{D}$. For this, we construct a new solution by adding a correction term to $G(z)^T$.

Write $F(z) = F_c + \phi_F(z)$, where $F_c = \{f_{c_1}, f_{c_2}, ...\} \in l^2$ and $\phi_F = \{\phi_{f_1}, \phi_{f_2}, ...\} \in \mathbb{I}_{l^2}$.

Using (3), we have that

$$I_{l^2} = (F(z)G(z)^T)I = G(z)^T F(z) + Q_{F(z)}Q^T_{G(z)}$$

This implies that

$$I_{l^2} = G(z)^T F_c + Q_{F(z)}Q^T_{G(z)} + G(z)^T \phi_F(z). \tag{4}$$

Applying $F_c^\star$ to (4), we get

$$F_c^\star = G(z)^T F_c F_c^\star + Q_{F(z)}Q^T_{G(z)}F_c^\star + G(z)^T\phi_F(z)F_c^\star.$$

Also, from Lemma 2.2, we know that $\|F_c\|^2 > 0$, so

$$\frac{F_c^\star}{\|F_c\|^2} = G(z)^T + Q_{F(z)}Q^T_{G(z)}\frac{F_c^\star}{\|F_c\|^2} + G(z)^T\phi_F(z)\frac{F_c^\star}{\|F_c\|^2}.$$

Thus,

$$G(z)^T + Q_{F(z)}Q^T_{G(z)}\frac{F_c^\star}{\|F_c\|^2} = \frac{F_c^\star}{\|F_c\|^2} - G(z)^T\phi_F(z)\frac{F_c^\star}{\|F_c\|^2}. \tag{5}$$

Define

$$U(z)^T := G(z)^T + Q_{F(z)}Q^T_{G(z)}\frac{F_c^\star}{\|F_c\|^2}.$$

Using (2), we can clearly see that

$$F(z)U(z)^T = F(z)G(z)^T + F(z)Q_{F(z)}Q^T_{G(z)}\frac{F_c^\star}{\|F_c\|^2} = F(z)G(z)^T = 1, \text{ for all } z \in \mathbb{D}.$$

Also, the right side of (5) shows that the solution U is in $(H^\infty_{\mathbb{I}}(\mathbb{D}))_{l^2}$.

For the norm estimate, we have that $\|U\|_\infty \leq \left(1 + \frac{1}{\|F_c\|}\right)\|G\|_\infty$.

Hence,

$$\|U\|_\infty \leq \left(1 + \frac{1}{\|F_c\|}\right)\frac{9}{\epsilon^2}\ln\left(\frac{1}{\epsilon^2}\right).$$

This completes the proof of Theorem 1. □

Proof of Theorem 1.2. Let $F \in H^\infty_{\mathbb{I}}(\mathbb{D})_{l^2}$, and suppose

$$|h(z)| \leq F(z)F(z)^*\psi\left(F(z)F(z)^*\right) \leq 1 \text{ for all } z \in \mathbb{D}$$

By Treil's theorem, there exists $G \in H^\infty_{l^2}(\mathbb{D})$ such that

$$F(z)G(z)^T = h(z) \quad \text{for all} \quad z \in \mathbb{D}$$

and $\|G\|_\infty \leq C_0$, where C_0 is the estimate for the $H^\infty(\mathbb{D})$-solution obtained in [22].

Writing $F(z) = F_c + \phi_F(z), h(z) = h_c + \phi_h(z)$ and using the relation (3) as in the proof of Theorem 1.1, we get

$$h_c \frac{F_c^*}{\|F_c\|^2} + \left(\phi_h - G(z)^T \phi_F(z)\right) \frac{F_c^*}{\|F_c\|^2} = G(z)^T + Q_{F(z)} Q_{G(z)}^T \frac{F_c^*}{\|F_c\|^2}. \quad (6)$$

Define

$$V(z)^T := G(z)^T + Q_{F(z)} Q_{G(z)}^T \frac{F_c^*}{\|F_c\|^2}$$

It's clear that

$$F(z)V(z)^T = h(z), \text{ for all } \ z \in \mathbb{D}.$$

Since $G \in (H^\infty(\mathbb{D}))_{l^2}$ and the elements of ϕ_F are in $\mathbb{I}$, the left side of the equation (6) shows that the solution V is in $(H^\infty_{\mathbb{I}}(\mathbb{D}))_{l^2}$.

As in the corona theorem, for the norm estimate, we have that $\|V\|_\infty \leq \left(1 + \frac{1}{\|F_c\|}\right) \|G\|_\infty \leq C_0 \left(1 + \frac{1}{\|F_c\|}\right)$, where C_0 is the norm of the $H^\infty(\mathbb{D})$ solution, G, obtained in [22].

□

Proof of Corollary 1. The proof of this corollary follows similarly as the proof of Theorem 1.2 by using Wolff's Theorem instead of Treil's Theorem. □

Acknowledgement: The author would like to thank the reviewer for the thorough and constructive review, which improved the over-all presentation of this paper significantly. Also, the author would like to thank T. Trent for his helpful comments.

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